Determinants & Matrices

BASIC CONCEPT Section - 1

1.1 Introduction (Second Order)

The following pattern represents a second order determinant.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The four numbers a_1, a_2, b_1 and b_2 are called the elements of the determinant. The elements in the horizontal line are said to form a row and the elements in the vertical line are said to form a column of the determinant. The above determinant is 2^{nd}-order as it contains 2 rows and 2 columns.

$$\begin{array}{cc} & \text{Columns} \\ & \downarrow \quad \downarrow \\ \text{Rows} \begin{array}{c}\rightarrow \\ \rightarrow\end{array} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{array}$$

1.2 Value of the Determinant

The value of 2^{nd}-order determinant is given as :

$$\text{value} = a_1 b_2 - a_2 b_1$$

THIRD-ORDER DETERMINANT Section - 2

2.1 Introduction

A determinant which consists of 3 rows and 3 columns is called a 3^{rd}-order-determinant and is of the following form.

$$\begin{array}{ccc} & & \text{Columns} \\ & & C_1 \quad C_2 \quad C_3 \\ & & \downarrow \quad \downarrow \quad \downarrow \\ \text{Rows} & \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ R_3 \rightarrow \end{array} & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \end{array} \qquad \begin{array}{l} C : \text{Columns} \\ R : \text{Rows} \end{array}$$

Elements of a determinant are denoted by : a_{ij} where i : represents Row No., j : represents column No.

Let us consider the following third order determinant.

$$\begin{vmatrix} 2 & -4 & 6 \\ 0 & 1 & 2 \\ -1 & 5 & 3 \end{vmatrix} \qquad \begin{array}{l} a_{11} = \text{element in 1st row and 1st column} = 2 \\ a_{32} = \text{element in 3rd row and 2nd column} = 5 \\ a_{33} = \text{element in 3rd row and 3rd column} = 3 \end{array}$$

Hence in general, a third order determinant can be represented as follows :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2.2 Minors and Cofactors

Minor of an element :

If we take an element of the determinant and delete (remove) the row and column containing that element, the determinant of the elements left is called the minor of that element. It is denoted by M_{ij}.

Consider the determinant :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of $a_{11} = M_{11}$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Minor of $a_{22} = M_{22}$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Similarly you can see yourself that :

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{and} \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor of an element :

The cofactor of an element a_{ij} (i.e., the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij}.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note : Clearly, we see that, it we apply the appropriate sign to the minor of an element, we have its cofactor.
The signs form a check-board pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

2.3 Method to evaluate the 3rd-order determinant

Consider the following determinant :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expansion of determinants using cofactors

A determinant can be evaluated by taking elements of any row or column and multiplying with their cofactors. Consider the following determinant :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding determinant by first row and taking appropriate signs.

(remember the checker -board pattern)

$$D = +a_1 M_{a_1} - b_1 M_{b_1} + c_1 M_{c_1}$$

$$= +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

We can also expand the determinant by 1st column taking appropriate signs.

$$D = +a_1 M_{a_1} - a_2 M_{a_2} + a_3 M_{a_3}$$

$$= +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

***Illustrating the Concepts* :**

Evaluate : $\begin{vmatrix} 3 & 2 & 12 \\ 0 & 1 & 8 \\ 2 & 9 & 7 \end{vmatrix}$.

Expanding The Determinant By 1st Column, we get :

$$D = 3 \times \begin{vmatrix} 1 & 8 \\ 9 & 7 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 12 \\ 9 & 7 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 12 \\ 1 & 8 \end{vmatrix}$$

$\Rightarrow \quad D = 3\,(7 - 72) - 0 + 2\,(16 - 12)$

$\Rightarrow \quad D = -187$

PROPERTIES AND THEOREMS OF DETERMINANTS — Section - 3

3.1 Properties

Determinants have some properties that are useful as they permit to generate equal determinants with different and simple configuration of entries. This in turn, helps us to find value of determinants. In other words, they help us in their transformations.

We shall list these properties below and give their proofs using third order determinants for any order.

(i) If rows be changed into columns and columns into rows, the determinant remains unaltered.

Proof : Let us consider the determinant :

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Evaluating by Ist row, we get :

$$D = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1) \quad \text{. . . (i)}$$

If D' be the determinant obtained by changing rows into columns and columns into rows :

Evaluating by Ist column, we get : $D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D' = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1) \quad \text{. . . (ii)}$$

Using (i) and (ii), D = D′

(ii) If any two rows (or columns) of a determinant are interchanged, the resulting determinant is the negative of the original determinant.

Proof :

Let D be original determinant (same as above). Now, Let D' be the determinant obtained by interchanging the first and second rows of D.

$$D' = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Evaluating by 1st row, we get :

$$\begin{aligned} D' &= b_1 (a_2c_3 - a_3c_2) - b_2 (a_1c_3 - a_3c_1) + b_3 (a_1c_2 - a_2c_1) \\ &= b_1 a_2c_3 - b_1a_3c_2 - b_2a_1c_3 + a_3c_1b_2 + b_3a_1c_2 - b_3a_2c_1 \\ &= a_1 (b_3c_2 - b_2c_3) - a_2 (b_3c_1 - b_1c_3) + a_3 (b_2c_1 - b_1c_2) \\ &= -[a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)] \\ &= -D \end{aligned}$$

Note : If any line of a determinant D be passed over 'm' parallel lines, the resulting determinant D' is equal to $(-1)^m D$.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Rightarrow \quad D' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

$$\Rightarrow \quad D' = (-1)^2 D = D$$

(iii) If two rows (or two columns) in a determinant have corresponding entries equal, the value of determinant is equal to zero.

Proof :

Let determinant D has 2^{nd} and 3^{rd} rows identical. i.e.,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Let D' be the determinant obtained by interchanging the second & third row :

$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Clearly, the determinant D' remains same as D, but from property 2, its value is $-D$.

$\Rightarrow \quad D = -D$

$\Rightarrow \quad 2D = 0 \quad \Rightarrow \quad D = 0$

(iv) If each of the entries of one row (or column) of a determinant is multiplied by k, then the determinant is multiplied by k.

Proof :

Let D be the original determinant and D' be the determinant obtained from D by multiplying the elements of first column by k.

$$D' = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$$

Evaluating D' by first column, we get :

$$D' = ka_1 (b_2c_3 - c_2b_3) - ka_2 (b_1c_3 - c_1b_3) + ka_3 (b_1c_2 - c_1b_2)$$

$$D' = k [a_1 (b_2c_3 - c_2b_3) - a_2 (b_1c_3 - c_1b_3) + a_3 (b_1c_2 - c_1b_2)]$$

$$D' = k (D)$$

(v) If each entry in a row (or column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants.

Proof :

Let $$D = \begin{vmatrix} a_1 + P_1 & b_1 & c_1 \\ a_2 + P_2 & b_2 & c_2 \\ a_3 + P_3 & b_3 & c_3 \end{vmatrix}$$

Evaluating D, by first column, we get :

$$D = (a_1 + P_1) (b_2c_3 - c_3b_2) - (a_2 + P_2) (b_1c_3 - c_1b_3) + (a_3 + P_3) (b_1c_2 - c_1 b_2)$$

$$= a_1 (b_2c_3 - c_3b_2) - a_2 (b_1c_3 - c_1b_3) + a_3 (b_1c_2 - c_1b_2) + P_1 (b_2c_3 - c_3b_2) - P_2 (b_1c_3 - c_1b_3) + P_3 (b_1c_2 - c_1b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} P_1 & b_1 & c_1 \\ P_2 & b_2 & c_2 \\ P_3 & b_3 & c_3 \end{vmatrix}$$

Hence, $$\begin{vmatrix} a_1+P_1 & b_1 & c_1 \\ a_2+P_2 & b_2 & c_2 \\ a_3+P_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} P_1 & b_1 & c_1 \\ P_2 & b_2 & c_2 \\ P_3 & b_3 & c_3 \end{vmatrix}$$

(vi) If to each element of a line (row or column) of a determinant be added the equi-multiples of the corresponding elements of one or more parallel lines, the determinant remains unaltered.

$$\begin{vmatrix} a_1+la_2+ma_3 & a_2 & a_3 \\ b_1+lb_2+mb_3 & b_2 & b_3 \\ c_1+lc_2+mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof :

Let $$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

D' be the determinant obtained by adding l times the elements of 2^{nd} column and m times the elements of 3^{rd} column to the corresponding elements of 1^{st} column of D.

$$\Rightarrow \quad D' = \begin{vmatrix} a_1+la_2+ma_3 & a_2 & a_3 \\ b_1+lb_2+mb_3 & b_2 & b_3 \\ c_1+lc_2+mc_3 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} la_2 & a_2 & a_3 \\ lb_2 & b_2 & b_3 \\ lc_2 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ma_3 & a_2 & a_3 \\ mb_3 & b_2 & b_3 \\ mc_3 & c_2 & c_3 \end{vmatrix}$$ [By property (v)]

$$\Rightarrow \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + l\begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix} + m\begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix}$$ [By property (iv)]

$$\Rightarrow \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + l\,(0) + m\,(0)$$ [By property (iii)]

$$\Rightarrow \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(vii) If each entry in any row (or any column) of a determinant is zero, then the value of determinant is equal to zero.

Proof :

Let $$D = \begin{vmatrix} a_1 & 0 & b_1 \\ a_2 & 0 & b_2 \\ a_3 & 0 & b_3 \end{vmatrix}$$

Clearly, evaluating by 2nd column, $D = 0$

(viii) If the elements of a determinant that involve x are polynomials in x, and if the determinant is equal is zero when a is substituted for x, then $x - a$ is a factor of given determinant.

To clearly illustrate this property, we will consider an example.

Illustrating the Concepts :

Prove that $D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ac & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

If a be substituted for b, first two columns become identical therefore force by property 3, $D = 0$.

Thus $(a - b)$ is a factor of determinant.

Similarly, $(b - c)$ and $(c - a)$ are also factors of determinant. [By property (viii)]

Now, since given determinant is homogenous and symmetrical in a, b and c and of fifth degree, there must be another factor which should be quadratic and symmetrical in a, b and c.

Let $D = (a - b)(b - c)(c - a)[k_1(a^2 + b^2 + c^2)$

$+ k_2(ab + ac + bc)]$ . . . (i)

Now determine k_1 and k_2

Put $a = 0$, $b = 1$, $c = 2$ on both sides of (i), we get :

$4 = 2(5k_1 + 2k_2)$ . . . (ii)

Now put $a = 0$, $b = 2$, $c = 3$, in (i) we get :

$36 = 6(13k_1 + 6k_2)$ . . . (iii)

Solving (ii) and (iii)

$k_1 = 0$ and $k_2 = 1$

$\Rightarrow$ $D = (a - b)(b - c)(c - a)(ab + bc + ca) =$ R.H.S.

3.2 Theorems

(i) The sum of the products of the elements of any row (or column) of a determinant with the corresponding co-factors is equal to the value of determinant.

Let us consider a determinant D :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

According to the theorem ; (taking 1st row)

$$a_1 C_{a_1} + a_2 C_{a_2} + a_3 C_{a_3} = D$$

From L.H.S.

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \times (-1) \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= D = \text{R.H.S.}$$

(ii) The sum of the products of the elements of the row (or column) with the co-factors of the corresponding elements of any other row (or column) is zero.

Let $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

According to the theorem ; $a_1 C_{b_1} + a_2 C_{b_2} + a_3 C_{b_3} = 0$

from L.H.S.

$$a_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + a_2(-1)\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 = \text{R.H.S.}$$ [By property (iii)]

3.3 An Important Note

Try to understand meaning of following operations.

C : column, R : row

$C_1 \to C_1 - C_2$: Writing C_1 (subtracting 2nd column from 1st column)

$R_1 \to R_1 - R_2$: Writing R_1 (subtracing 2nd row from first row)

$R_2 \to R_2 - 2R_3$: Writing R_2 (subtracting the twice of 3rd row from second row)

- Try to get all the elements of any row or column as 1 :

$$\Rightarrow \begin{vmatrix} 1 & \dots & \dots \\ 1 & \dots & \dots \\ 1 & \dots & \dots \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

- Then apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

  or $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

- In this way, 2 elements of C_1 or R_1 will be equal to zero. Then expand the determinant using C_1 or R_1.

Illustration - 1

The value of $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ *is* :

(A) 0 **(B)** 1 **(C)** –1 **(D)** *None*

SOLUTION : (A)

If we simply try to evaluate the determinant by any row or column, lot of calculations will be involved. So in order to make things simpler, we will apply property 6 as follows:

Operate $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$D = \begin{vmatrix} 265-240 & 240-219 & 219 \\ 240-225 & 225-198 & 198 \\ 219-198 & 198-181 & 181 \end{vmatrix} = \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

Operate $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - 10\,C_2$

$$\Rightarrow D = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix} = 4\begin{vmatrix} 1 & 21 & 9 \\ -3 & 27 & -72 \\ 1 & 17 & 11 \end{vmatrix}$$

[Property-(iv)]

Operate $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow D = 4\begin{vmatrix} 1 & 21 & 9 \\ 0 & 90 & -45 \\ 0 & -4 & 2 \end{vmatrix}$$

Now evaluating by 1st column, to get

$$D = 4\,[1\,(180 - 180) - 0\,(42 + 36) + 0\,(-45 \times 21 - 9 \times 90)] = 0$$

Note : While evaluating a determinant, try to make at least two elements of either a row or a column as zero and then it becomes easy to open a determinant by that row or column (as done is last Ex.)

Illustration - 2

The value of $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ *is :*

(A) 1 **(B)** 0 **(C)** $a+b$ **(D)** $a-b$

SOLUTION : (B)

Operate $C_3 \to C_3 + C_2$ in L.H.S. to get :

$$\text{L.H.S.} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = (a+b+c)\,(0)$$

$= 0 =$ R.H.S. [Using property (iii) and (iv)]

Illustration - 3

The value of $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ $(\omega : \textit{cube root of unity})$ *is :*

(A) 1 **(B)** -1 **(C)** 0 **(D)** ω

SOLUTION : (C)

Operate $C_1 \to C_1 + C_2 + C_3$ to get :

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix}$$ [sum of cube roots of unity]

$= 0$ [Using property (vii)]

Illustration - 4

The value of $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ *is :*

(A) a **(B)** b **(C)** 0 **(D)** *None of these*

SOLUTION : (C)

Operate $R_1 \to R_1 - R_2$ and then $R_2 \to R_2 - R_3$ to get :

$$= \begin{vmatrix} a-b & a-b & a-b \\ b-c & b-c & b-c \\ c-a & c-b & 0 \end{vmatrix}$$

$$= (a-b)\,(b-c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ c-a & c-b & 0 \end{vmatrix} = 0$$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS USING DETERMINANTS Section - 4

Method of solving system of Linear equations

Consider a system of simultaneous linear equations in three variables namely x, y, z

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- Elements of D are arranged in the same order as they occur as coefficients in the equations

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

- D_1 is obtained by replacing Ist column of D by d_1, d_2 and d_3

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

- D_2 is obtained by replacing IInd column of D by d_1, d_2 and d_3.

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

- D_3 is obtained by replacing IIIrd column of D by d_1, d_2 and d_3

The following cases can arise :

(A) (i) $D \neq 0$: In such case, the system has precisely one solution (unique solution), which is given by Cramer's rule :

$$x = \frac{D_1}{D},\ y = \frac{D_2}{D},\ z = \frac{D_3}{D}$$

(ii) $D = 0$ and at least one of the determinants D_1 D_2 or D_3 is non-zero, then the system is inconsistent i.e., it has no solution.

(iii) $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system has infinite solutions.

(B) Homogenous and Non-Homogenous System

(i) If $d_1 = d_2 = d_3 = 0$, then system is known as a homogenous system of equations. If the system of equations is homogenous, then $D_1 = D_2 = D_3 = 0$ ($\because$ value of determinant is zero, if one column has all elements = 0)

$x = y = z = 0$ and non-trivial solution (infinite solutions) exists if and only if $D = 0$.

The system has at least the trivial solution, i.e., $x = y = z = 0$.

(ii) If at least one of the d_1 ,d_2 and d_3 is non-zero, the system is known as non-homogenous system.

(C) An Important Theorem

A system of three linear equations in two variables i.e.,

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

is concurrent if : $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Illustrating the Concepts :

Investigate for what values of λ and μ, the following system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) a unique solution (ii) no solution and (iii) an infinite number of solutions.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \lambda - 3,\ D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 2\lambda - 16 + \mu,\ D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 2(2\lambda - \mu + 4),\ D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = \mu - 10$$

$\lambda \neq 3 \Rightarrow$ D $\neq 0$. Hence the given system has unique solution

$\lambda = 3 \Rightarrow$ D $= 0$ and $\mu \neq 10 \Rightarrow D_1 \neq 0, D_2 \neq 0, D_3 \neq 0$. Hence the given system has no solution.

$\lambda = 3 \Rightarrow$ D $= 0$ and $\mu = 10 \Rightarrow D_1 = 0, D_2 = 0, D_3 = 0$. Hence the given system has infinite solutions.

Illustrating the Concepts :

Show that the equations :

$$x + y + z = 3$$
$$3x - 5y + 2z = 8$$
$$5x - 3y + 4z = 14$$

are consistent and solve them

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} 3 & 1 & 1 \\ 8 & -5 & 2 \\ 14 & -3 & 4 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 5 & 14 & 4 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 1 & 1 & 3 \\ 3 & -5 & 8 \\ 5 & -3 & 14 \end{vmatrix} = 0$$

Hence the given system of equations is consistent and has infinite solutions.

Let $z = k$ where k is any arbitrary constant.

$\Rightarrow \quad x + y + k = 3$ and $3x - 5y + 2k = 8$

Solve the above equations to get :

$$x = \frac{23 - 7k}{8} \quad y = \frac{1 - k}{8} \text{ and } z = k$$

where k is any arbitrary constant.

> **Note :** If we observe carefully, subtracting (iii) equation from (i) equation will generate (ii) equation (Hence dependent equations)

Illustration - 5 *The value of k do the following homogenous system of equations possess a non-trivial solution?*

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$

(A) 1 (B) 33/4 (C) 33/2 (D) *None of these*

SOLUTION : (B)

For non-trivial solutions, $D = 0$

$$\Rightarrow \quad D = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

Operate $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$ to get :

$$\begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix} = 0$$

Evaluating by 1st column, we get, $k = \dfrac{33}{2}$

Illustration - 6 *The solution for the following system of equations.*

$$x + 4y - 2z = 3$$
$$3x + y + 5z = 7$$
$$2x + 3y + z = 5 \text{ is :}$$

(A) *Consistent* (B) *Unique* (C) *No solution* (D) *None of these*

SOLUTION : (C)

For a non-homogenous system (like the given one), there is no solution if $D = 0$ and at least one out of D_1, D_2, D_3 is non-zero.

$$D = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix}$$

Evaluating D, we get

$$D = 1\,(1 - 15) - 3\,(4 + 6) + 2\,(20 + 2)$$

$$\Rightarrow \quad D = 0$$

$$\text{Now} \quad D_1 = \begin{vmatrix} 3 & 4 & -2 \\ 7 & 1 & 5 \\ 5 & 3 & 1 \end{vmatrix} = -2 \quad \Rightarrow \quad D_1 \neq 0$$

Hence the system has no solution or we can say that the system is inconsistent.

Illustration - 7 *The value of 'λ' for which the set of equations*

$$x + y - 2z = 0$$
$$2x - 3y + z = 0$$
$$x - 5y + 4z = \lambda \quad \text{are consistent.}$$

(A) 1 (B) 2 (C) 0 (D) –1

SOLUTION : (C)

A non-homogenous system has unique solution if $D \neq 0$ and infinite solutions if $D = D_1 = D_2 = D_3 = 0$

$$\text{Now} \quad D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ 1 & -5 & 4 \end{vmatrix} = 0$$

Hence D_1, D_2 and D_3 should all be zero.

i.e. $D_1 = 0$

$$\Rightarrow \quad D_1 = \begin{vmatrix} 0 & 1 & -2 \\ 0 & -3 & 1 \\ \lambda & -5 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda\,(1 - 6) = 0 \quad \Rightarrow \quad \lambda = 0$$

Illustration - 8 *System the equations :*

$$4x + 5y - 9z = 0$$
$$11x - 4y - 7z = 0$$
$$x + 2y - 3z = 0 \text{ have}$$

(i.e. satisfied by the same values of x,y and z)

(A) *Consistent* **(B)** *Inconsistent* **(C)** *No solution* **(D)** *None*

SOLUTION : (A)

If a homogenous system of equations is consistent, D must be zero, because D_1, D_2 and D_3 are already zero.

$$\Rightarrow \quad D = \begin{vmatrix} 4 & 5 & -9 \\ 11 & -4 & -7 \\ 1 & 2 & -3 \end{vmatrix}$$

$\Rightarrow \quad D = 4\,(12 + 14) - 5\,(-33 + 7) - 9\,(22 + 4)$

$\Rightarrow \quad D = 104 + 130 - 234$

$\Rightarrow \quad D = 0$

Hence the equations are consistent and have infinite solutions.

Matrices

BASICS Section - 5

5.1 Definition

A matrix is a rectangular structure in which an array of numbers is written within brackets. These numbers may be real or complex. In general, a matrix is usually represented by a capital letter and classified by its dimensions. It may be represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{bmatrix}_{m \times n} \quad \text{or} \quad A = (a_{ij})_{m \times n}$$

where the first suffix in a_{ij} namely 'i' denotes the row number in which a_{ij} lies and the second suffix 'j' denotes the column number in which the element a_{ij} lies.

A matrix with m rows and n columns is called $m \times n$ matrix and the size (or dimension) of this matrix is said to be $m \times n$. $m \times n$ is also known as the order of the matrix.

Note : *The matrix is not a number. It has no numerical value. But it is a structure.*

Consider the following information regarding the number of men and women workers in three factories,

Factories	*Men workers*	*Women workers*
I	30	5
II	25	11
III	27	6

Illustrating the Concepts **:**

Represent the above information in the form of 3 × 2 matrix. What does the entry in the third row and second column represent ?

The information is represented in the form of a 3 × 2 matrix as follows : $\begin{bmatrix} 30 & 5 \\ 25 & 11 \\ 27 & 6 \end{bmatrix}$

The entry in the third row and second column represents the number of women workers in factory III.

5.2 Important Terms Related to Matrices

(i) Element of Matrix :

Each of the mn numbers of an $m \times n$ matrix is called an element.

(ii) Leading Element :

The element lying in the first row and first column is called leading element (or leading entry) of a matrix.

(iii) Diagonal Elements :

An element of a matrix $A = [a_{ij}]$ is said to be diagonal element if $i = j$. Thus an element whose row suffix equals to the column suffix is a diagonal element. *e.g.* $a_{11}, a_{22}, a_{33} \ldots$ are all diagonal elements.

(iv) Principal Diagonal :

The line along which the diagonal elements lie is called the principal diagonal or simply the diagonal of the matrix.

TYPES OF MATRICES — Section - 6

6.1 (i) Row Matrix

The matrix having order $1 \times n$ or matrix having only one row is called row matrix. In row matrix, the number of columns may be 'n' where $n \in N$.

Example : (i) $[9\ \ 5\ \ 2]$ (ii) $[5,\ 8, -1, 2]$ (iii) $[b_1, b_2, b_3, \ldots\ldots b_n]$

(ii) Column Matrix

The matrix having order $m \times 1$ or matrix having only one column is called column matrix. In column matrix the number of rows may be 'n' where 'n' $\in N$.

Example : (i) $\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 \\ 9 \\ -1 \\ -3 \end{bmatrix}$ (iii) $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$

(iii) Zero Matrix or Null Matrix

A matrix each of whose elements is zero is called a zero matrix or null matrix. A zero matrix of order $m \times n$ is denoted by $O_{m \times n}$.

Example : (i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{3\times 2}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{2\times 3}$

(iv) Square Matrix

A matrix in which the number of rows is equal to the number of columns is called a square matrix, otherwise, it is said to a rectangular matrix. Thus, a matrix $A = [a_{ij}]_{m \times n}$ is said to be a square matrix if $m = n$ and a rectangular matrix if $m \neq n$.

(v) Diagonal Matrix

A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if all its non-diagonal elements are zero.

Thus $A = [a_{ij}]_{n \times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ are diagonal matrix.

An n-rowed diagonal matrix is briefly written as diagonal $(d_1, d_2, \ldots, d_n)$ where $d_1, d_2, \ldots d_n$ are the diagonal elements. Thus, the above two diagonal matrices can be written as diagonal (2, 3, 4) and diagonal (a, b, c) respectively.

Note : The diagonal elements of diagonal matrix may or may not be zero.

(vi) Unit Matrix or Identity Matrix

A square matrix is said to be a unit matrix or identity matrix if

(*i*) all its non-diagonal elements are zero,

(*ii*) all its diagonal elements are each equal to unity *i.e.* 1.

Thus $A = [a_{ij}]_{n \times n}$ is said to be a unit matrix or identity matrix if $a_{ij} = \begin{cases} 0 & \text{when} \quad i \neq j \\ 1 & \text{when} \quad i = j \end{cases}$

A unit matrix of order n is defined by I_n or simply by I.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are all unit matrix and denoted by I_2, I_3, I_4 respectively.

(vii) Scalar Matrix

A square matrix $A = [a_{ij}]$ is said to be a scalar matrix if

(i) all its non-diagonal elements are zero

(ii) all its diagonal elements are equal.

Thus $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0 & \text{when} \quad i \neq j \\ k & \text{when} \quad i = j \end{cases}$

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$

are scalar matrix. They can be written as diagonal (2, 2, 2) and diagonal (d, d, d, d) respectively.

(viii) Upper Triangular Matrix

A square matrix all of whose elements below the principal diagonal are zero is called an upper triangular matrix.

Thus $A = [a_{ij}]_{n \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

Example : $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix.

Note : The elements along the principal diagonal or above it may or may not be zero.

(ix) Lower Triangular Matrix

A square matrix all of whose elements above the principal diagonal are zero is called a lower triangular matrix.

Thus $A = [a_{ij}]_{n \times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

Example : $\begin{bmatrix} -1 & 0 & 0 \\ 5 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is a lower triangular matrix.

(x) Triangular Matrix :

A matrix which is either a lower triangular matrix or an upper triangular matrix is called a triangular matrix.

(xi) Triple diagonal Matrix :

A square matrix is triple diagonal matrix if all of its element except on principal diagonal and the diagonal lying above and below it are zero.

(xii) SUR or Trace :

The sum of all diagonal elements of a matrix is called Trace. This is defined only for a square matrix.

i.e., trace $= \sum a_{ij}$ when $i = j$

(xiii) Comparable Matrix :

Two matrix are said to be comparable when they are of the same type. Thus two matrices

$A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are comparable if $m = p$ and $n = q$.

(xiv) Equality of two matrix :

Two matrix A and B are said to be equal (written as $A = B$) if

(a) they are of the same type and

(b) their corresponding elements are equal.

Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$ then $A = B$ if

(a) $m = p,\ n = q$ (b) $a_{ij} = b_{ij}$ for all i and j.

ADDITION OF MATRICES Section - 7

7.1 Addition of matrix

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices. Only when they are of the same type, then their sum denoted by $(A + B)$ is a matrix of the same type $m \times n$, each of whose elements is obtained by adding the corresponding elements of matrix A and B.

Thus, if $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ then $A + B = \left[(a_{ij} + b_{ij})\right]_{m \times n}$, for all i and j

Note : If $A + B = C \equiv [c_{ij}]_{m \times n}$, then $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

***Illustrating the Concepts* :**

Given matrix : $A = \begin{bmatrix} 5 & -2 & 0 \\ 3 & 0 & 5 \\ -1 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 6 \\ 0 & 2 & 4 \end{bmatrix}$

Find (whichever defined).

(i) $A + B$ (ii) $A + D$.

SOLUTION :

(i) A is a matrix of the type 3×3.

B is a matrix of the type 3×2.

Since A and B are not of the same type.

$\therefore$ Sum $(A + B)$ is not defined.

(ii) Here A and D are matrix of the same type so the sum $(A + D)$ is defined and

$$A + D = \begin{bmatrix} 5 & -2 & 0 \\ 3 & 0 & 5 \\ -1 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5+4 & -2+1 & 0+2 \\ 3+1 & 0+5 & 5+6 \\ -1+0 & 0+2 & 8+4 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 2 \\ 4 & 5 & 11 \\ -1 & 2 & 12 \end{bmatrix}$$

7.2 Properties of matrix addition

(i) Addition of matrix is commutative.

i.e. if A and B are any two matrices conformable for addition (*i.e.* of the same type), then,

$$A + B = B + A$$

(ii) Addition of matrix is associative.

i.e. if A, B, C are matrices of the same type, then

$$(A + B) + C = A + (B + C)$$

(iii) Existence of additive identity.

i.e. for any matrix A, there exists the null matrix O of the same type such that

$$A + O = O + A = A.$$

(iv) Existence of additive inverse.

i.e. for any matrix A, there exists a unique matrix X of the same type such that

$$A + X = O = X + A.$$

(v) Cancellation Law : If A, B, C are three matrices of the same type, then

(a) $A + B = A + C \Rightarrow B = C$ (Left cancellation)

(b) $B + A = C + A \Rightarrow B = C$ (Right cancellation)

Note : For a given matrix $A = [a_{ij}]$, if there exists a unique matrix $[-a_{ij}]$ such that $[a_{ij}] + [-a_{ij}] = 0$.

This matrix $[-a_{ij}]$ is denoted by $-A$ and is called the additive inverse or negative of A. Thus, we have

$$A + (-A) = (-A) + A = 0$$

7.3 Difference of two matrix

Let A and B be two matrices of the same type. The subtraction of B from A, denoted by $A - B$, is the sum of A and the negative of B.

Thus $A - B = A + (-B)$

Illustration - 9

The value of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \sqrt{2} & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 4 & 5 & 6 \\ -2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 3 \\ 2 & 0 & 4 \\ \sqrt{2}-2 & 4 & 5 \end{bmatrix}$ *is :*

(A) $\begin{bmatrix} 0 & 1 & 1 \\ 3 & 5 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 0 & 2 & 2 \\ 3 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ **(C)** $\begin{bmatrix} 0 & -2 & 2 \\ 3 & -5 & -3 \\ 0 & 1 & 1 \end{bmatrix}$ **(D)** *None of these*

SOLUTION : (B)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \sqrt{2} & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 4 & 5 & 6 \\ -2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 3 \\ 2 & 0 & 4 \\ \sqrt{2}-2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 5 \\ 5 & 5 & 7 \\ \sqrt{2}-2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -2 & 0 & -4 \\ -\sqrt{2}+2 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+(-3) & 4+(-2) & 5+(-3) \\ 5+(-2) & 5+0 & 7+(-4) \\ \sqrt{2}-2+(-\sqrt{2}+2) & 4+(-4) & 5+(-5) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 \\ 3 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Illustration - 10

Given $A = \begin{bmatrix} 1 & 2 & -5 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$. *The matrix C such that* $A + 2C = B$ *is :*

(A) $C = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ **(B)** $C = \frac{1}{2}\begin{bmatrix} 0 & -3 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ **(C)** $C = \frac{1}{2}\begin{bmatrix} -1 & -3 & 0 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ **(D)** $C = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

SOLUTION : (A)

Given: $A + 2C = B$ and $2C = B - A$

$$\Rightarrow \quad 2C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & -1-2 & 2+3 \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0+1 & 3-1 \end{bmatrix}$$

$$\Rightarrow \quad 2C = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad C = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

7.4 Multiplication of a matrix by a scalar

Let $A = [a_{ij}]_{m \times n}$ be any matrix and α be any scalar, then multiplication of the matrix A by the scalar α is denoted by αA and is obtained by multiplying each element of A by α. Thus

$$\alpha A = \alpha \,.\, [a_{ij}]_{m \times n} = [\alpha a_{ij}]_{m \times n}$$

7.5 Properties of Multiplication by a scalar

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same type and α and β are any scalars, then

(i) $\alpha(A + B) = \alpha A + \alpha B$

(ii) $(\alpha + \beta) A = \alpha A + \beta A$

(iii) $\alpha(\beta A) = (\alpha\beta) A$

MATRIX MULTIPLICATION Section - 8

8.1 Conformability for multiplication

Two matrices A and B are said to be conformable for the product AB (in this very order of A and B) if the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor). Thus, if A and B are of the types $m \times n$ and $p \times q$ respectively, then

(i) AB is defined if number of columns in A = number of rows in B

$\Rightarrow$ if $n = p$.

(ii) BA is defined if number of columns in B = number of rows in A

$\Rightarrow$ if $q = m$.

8.2 Multiplication of Matrix

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices conformable for the product AB, then AB is defined as the matrix $C = [c_{ij}]_{m \times p}$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$ $\quad (i = 1, 2, \ldots\ldots, m\ ;\ j = 1, 2, \ldots\ldots, p)$

i.e. c_{ij} = sum of the products of the elements of i^{th} row of A with the elements of the j^{th} column of B.

Illustrating the Concepts :

Find the product of the matrices : $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & -5 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -5 & 6 \end{bmatrix}$.

Let the given matrices be denoted by A and B respectively.

A is of the order 2×3 and B is of the order 3×2.

$\therefore$ The product AB is defined and is a matrix of the order 2×2.

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2.1 + 3.3 + 4(-5) & 2.2 + 3(-4) + 4.6 \\ -1.1 + 2.3 + (-5)(-5) & -1.2 + 2(-4) + (-5).6 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 16 \\ 30 & -40 \end{bmatrix}$$

8.3 Properties of matrix multiplication

(i) Matrix multiplication is associative :

i.e. If A, B, C are matrices of the type $m \times n$, $n \times p$ and $p \times q$ respectively, then $(AB)\,C = A\,(BC)$

(ii) Multiplication of matrices is distributive with respect to the addition of matrix :

i.e. If A is a matrix of the type $m \times n$ and B and C are matrices both of the same type $n \times p$, then $A\,(B + C) = AB + AC$.

8.4 Positive Integral Powers of Matrix

Let A be any square matrix of order n.

Then $A^2 = A.A$

$A^3 = A.A.A$

and $A^m = A.A.A\ldots m$ times

All are square matrix of order n.

(i) $A^m.\,A^n = (A.A.A\ldots m \text{ times})\,(A.A.A\ldots n \text{ times})$

$= A.A.A\ldots(m+n)$ times

$= A^{m+n}$

Similarly, (ii) $(A^m)^n = A^{mn}$

Also, we define $A^0 = I$

Note : Matrix multiplication of matrix is not commutative in general *i.e.* $AB \neq BA$.

Illustration - 11

Let $f(x) = x^2 - 5x + 6$. *Then the value of* $f(A)$, *is if* $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ *is* :

(A) $\begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & -10 \\ 5 & 4 & 4 \end{bmatrix}$ **(B)** $\begin{bmatrix} 0 & 0 & -3 \\ 1 & 2 & 10 \\ 6 & 4 & 1 \end{bmatrix}$ **(C)** $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 4 & -5 & 4 \end{bmatrix}$

SOLUTION : (C)

$f(x) = x^2 - 5x + 6 = x^2 - 5x + 6.x^0$ $[\because x^0 = 1]$

$\therefore$ $f(A) = A^2 - 5A + 6A^0 = A^2 - 5A + 6I$ ... (i)

Now $A^2 = A.A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Substituting the values of A^2, A and I in (i), we get:

$$f(A) = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Note : Matrices are such systems where the product of two non-zero matrices can be a zero matrix.

If $A \neq 0$, $B \neq 0$ and $AB = 0$, then A and B are called zero divisors.

8.5 Transpose of a matrix

Given a matrix A, then the matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^T.

For example, if $A = \begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & -1 & 3 & 7 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 3 \\ 5 & 7 \end{bmatrix}$

Thus if A is of the type $m \times n$. then A' is of the type $n \times m$.

In symbols, If $A = [a_{ij}]_{m \times n}$, then

$A' = [\,a'_{ij}\,]_{n \times m}$ where $a'_{ij} = a_{ji}$ *i.e.* $(i, j)^{\text{th}}$ element of $A' = (j, i)^{\text{th}}$ element of A.

8.6 Properties of Transpose of matrix

If A', B' denote the transpose of A and B respectively, then

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$; provided A, B being conformable for addition.

(iii) $(kA)' = kA'$ where k is any scalar.

(iv) $(AB)' = B'A'$

In general $(A_1A_2A_3 \ldots\ldots A_{n-1}A_n)' = A'_n A'_{n-1} \ldots A_2' A_1'$

(v) If 'A' is an invertible matrix, then $(A^{-1})' = (A')^{-1}$

SPECIAL TYPES OF MATRICES — Section - 9

(i) Symmetric Matrix

A matrix A is said to be symmetric if $A' = A$ *i.e.* if the transpose of a matrix is equal to itself.

Let $A = [a_{ij}]_{m \times n}$

$\therefore$ $A' = [\alpha_{ij}]_{n \times m}$ where $\alpha_{ij} = a_{ij}$

$A = A'$ if and only if

(i) $m = n$ and (ii) $\alpha_{ij} = a_{ij}$

i.e. if A is a square matrix and $a_{ij} = a_{ji}$

Thus we may also define a symmetric matrix as :

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.

(ii) Skew-Symmetric Matrix

A matrix A is said to be skew-symmetric if $A' = -A$ *i.e.* when a matrix equals the negative of its transpose.

Now, let $A = [a_{ij}]_{m \times n}$

$\therefore$ $A' = [\alpha_{ij}]_{n \times m}$ where $\alpha_{ij} = -a_{ij}$.

Now $A' = -A$,

If $[\alpha_{ij}]_{m \times n} = [-a_{ij}]_{m \times n}$

or if $m = n$ and $a_{ji} = -a_{ij}$

or if A is a square matrix and $a_{ji} = -a_{ij}$

Thus we may also define a skew-symmetric matrix as:

A square matrix A is said to be skew symmetric if $a_{ij} = -a_{ji}$ for all i and j.

In particular, $a_{ii} = -a_{ii}$ when $j = i$

i.e. $2a_{ii} = 0$ for all i

or $a_{ii} = 0$

Note : Thus in a skew-symmetric matrix, all the diagonal elements must be zero.

Properties of symmetric and skew symmetric matrices:

(i) $\left.\begin{array}{c} A'A \\ AA' \\ A + A' \end{array}\right] \to$ Symmetric matrices

(ii) $A - A' \to$ Skew symmetric matrices

(iii) If A = skew symmetric then

A^2 = symmetric and A^3 = Skew symmetric *i.e.*

(a) If the power is even, then it is symmetric.

(b) If the power is odd, then it is skew symmetric.

(iv) If A is skew symmetric matrix of odd order then |A| = 0

If A is skew symmetric matrix of even order with integer elements, then |A| is perfect square.

Illustrating the Concepts :

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$. *Show that* $(AB)' = B'A'$.

We have $A' = \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 3 & 5 \end{bmatrix}$

and $B' = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}$

$\Rightarrow \quad AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1+2+6 & 3+0+12 \\ -4-2+10 & -12+0+20 \end{bmatrix}$

$\Rightarrow \quad AB = \begin{bmatrix} 9 & 15 \\ 4 & 8 \end{bmatrix}$

$\Rightarrow \quad (AB)' = \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix}$ . . . (i)

and $B'A' = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 3 & 5 \end{bmatrix}$

$= \begin{bmatrix} 9 & 4 \\ 15 & 8 \end{bmatrix}$ . . . (ii)

From (i) and (ii), we get: $(AB)' = B'A'$

***Illustrating the Concepts* :**

Show that any square matrix can be expressed as the sum of two matrices, one symmetric and the other anti-symmetric.

Let A be the given matrix.

Then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Now $(A + A')' = A' + (A')' = A' + A$

$\Rightarrow$ $A + A'$ is symmetric

$\Rightarrow$ $\frac{1}{2}(A + A')$ is symmetric.

Also $(A - A')' = A' - (A')' = A' - A = -(A - A')$

$\Rightarrow$ $A - A'$ is anti-symmetric.

$\Rightarrow$ $\frac{1}{2}(A - A')$ is anti-symmetric

$\therefore$ $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

= symmetric matrix + anti-symmetric matrix

Note : Anti-symmetric and skew-symmetric matrix are same.

(iii) Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

(iv) Periodic Matrix

A square matrix A is called periodic if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be period of A. For $k = 1$, we get $A^2 = A$ and we called it to be idempotent matrix.

(v) Nilpotent Matrix

A square matrix A is called Nilpotent matrix of order k provided it satisfies the relation $A^k = O$ and $A^{k-1} \neq O$, where k is positive integer and O is null matrix and k is the order of nilpotent matrix A.

(vi) Involutory Matrix

A square matrix A is called Involutory provided it satisfies the relation $A^2 = I$, where I is identity matrix.

(vii) Orthogonal Matrix

A square matrix A is called an orthogonal matrix if the product of the matrix A and its transpose A' is an identity matrix.

$$AA' = I$$

Note :
(i) If $AA' = I$ then $A^{-1} = A'$
(ii) If A and B are orthogonal, then AB is also orthogonal.
(iii) All above properties are defined for square matrix only.

Illustrating the Concepts **:**

Verify that $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix}$ *is an orthogonal matrix.*

Given :

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix} \text{ and } A' = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\text{Now } AA' = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence A is orthogonal matrix.

ADJOINT AND INVERSE OF SQUARE MATRIX — Section - 10

10.1 Adjoint of Square Matrix

The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factor in $|A|$.

In notation : If $A = [a_{ij}]_{n \times n}$, then adjoint of A, briefly written as $adj\,(A)$ is given by

Adjoint $A = [A_{ij}]'_{n \times n}$, where A_{ij} is the co-factor of a_{ij} in $|A|$.

Properties of Adjoint :

(i) $adj\,(O) = O$

(ii) $adj\,(I) = I$

(iii) adj (scalar) = scalar

(iv) adj (diagonal) = (diagonal)

(v) $adj\,(A^T) = (adj\,A)^T$

(vi) A is symmetric, then $adj\,(A)$ is symmetric

(vii) $adj\,(\lambda A) = \lambda^{n-1}\,adj\,A$, where n is order of matrix A

(viii) If A is Skew symmetric matrix of order 'n', then $adj\ A$ is

(a) Skew symmetric when n is even (b) Symmetric when n is odd

(ix) $adj\,(AB) = (adj\,B) \cdot (adj\,A)$

(x) $A\,(adj\,A) = (adj\,A)\,A = |A|\,I_n$ where I_n is unit matrix of order n

(xi) $|adj\,A| = |A|^{n-1}$

(xii) $adj\,(adj\,A) = |A|^{n-2} . A$ (A is non-singular matrix)

(xiii) $|adj\,(adj\,A)| = |A|^{(n-1)^2}$ (A is non-singular matrix)

(xiv) A and $adj\,A$ behave alike i.e.

(a) If A is singular, then adjoint of A is singular.

(b) If A is non-singular, then adjoint of A is non-singular.

(c) If A is invertible, then adjoint of A is also invertible.

Illustrating the Concepts :

Calculate the adjoint of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}$

$\therefore$ Co-factor of $(1, 1)^{th}$ element

i.e. $1 \equiv (-1)^2 \times (45 - 49) = -4$

Co-factor of $(1, 2)^{th}$ element

i.e. $2 \equiv (-1)^3 \times (27 - 28) = 1$

Co-factor of $(1, 3)^{th}$ element

i.e. $3 \equiv (-1)^4 \times (21 - 20) = 1$

Co-factor of $(2, 1)^{th}$ element

i.e. $3 \equiv (-1)^3 \times (18 - 21) = 3$

Co-factor of $(2, 2)th$ element

i.e. $5 \equiv (-1)^4 \times (9 - 12) = -3$

Co-factor of $(2, 3)^{th}$ element

i.e. $7 \equiv (-1)^5 \times (7 - 8) = 1$

Co-factor of $(3, 1)^{th}$ element

i.e. $4 \equiv (-1)^4 \times (14 - 15) = -1$

Co-factor of $(3, 2)^{th}$ element

i.e. $7 \equiv (-1)^5 \times (7 - 9) = 2$

Co-factor of $(3, 3)^{th}$ element

i.e. $9 \equiv (-1)^6 \times (5 - 6) = -1$

$\therefore$ adjoint $A = \begin{bmatrix} -4 & 1 & 1 \\ 3 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}' = \begin{bmatrix} -4 & 3 & -1 \\ 1 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

10.2 Inverse of Matrix

Before understanding the meaning of inverse of matrix, we must learn two types of matrices.

(i) Singular Matrix

A square matrix 'A' is said to be singular if $|A| = 0$.

(ii) Non-Singular Matrix

A square matrix 'A' is said to be non-singular if $|A| \neq 0$.

Inverse of a Matrix

Let A be an n-rowed square matrix. If there exists an n-rowed square matrix B such that

$$AB = BA = I_n.$$

then the matrix A is said to be invertible and B is called the inverse of A or reciprocal of A.

Note : 1. Only square matrices are invertible, *i.e.,* possess inverse.

2. From the definition of inverse given above, it follows that if B is the inverse of A, then A is inverse of B.

3. The necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$ (*i.e.* A is non-singular).

Finding the Inverse of Matrix Using Adjoint Matrix

We know that

$$A\,.\,(Adj\ A) = |A|\ I \quad \text{or} \quad \frac{A\,.\,(Adj\ A)}{|A|} = I \text{ provided } |A| \neq 0$$

Also $AA^{-1} = I \Rightarrow A^{-1} = \dfrac{(Adj.A)}{|A|}$

Illustrating the Concepts :

Compute the inverse of the matrix : $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

$\therefore$ $|A| = 1(0+25) - 3(0+10) - 2(-15) = 25$

$\Rightarrow$ A is non-singular $\Rightarrow$ A^{-1} exists.

$$adj\,A = \begin{bmatrix} 25 & -10 & -15 \\ -10 & 4 & 1 \\ -15 & 11 & 9 \end{bmatrix}' = \begin{bmatrix} 25 & -10 & -15 \\ -10 & 4 & 11 \\ -15 & 1 & 9 \end{bmatrix}$$

$$A^{-1} = \frac{adj.A}{|A|} = \frac{1}{25}\begin{bmatrix} 25 & -10 & -15 \\ -10 & 4 & 11 \\ -15 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

10.3 Properties of Inverse of a Matrix

(i) Inverse of a matrix if it exists is unique.

(ii) $AA^{-1} = A^{-1}A = I_n$

(iii) $(A^{-1})^{-1} = A$

(iv) $(kA)^{-1} = k^{-1}A^{-1}$ if $k \neq 0$.

(v) $(AB)^{-1} = B^{-1}A^{-1}$ in general $(ABC\,.\,.\,.\,.\,.\,.\,.\,Z)^{-1} = Z^{-1}\,Y^{-1}\,.\,.\,.\,.\,.\,.\,.\,.\,B^{-1}A^{-1}$

(vi) $(A^T)^{-1} = (A^{-1})^T$

(vii) If $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then A^{-1} exists if $\lambda_i \neq 0\ \forall\ i$ and $A^{-1} = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$.

Also, $A^m = \text{diag}\left(\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m\right)$ if $m \in N$

(viii) If a square matrix A satisfies the equation $a_0 + a_1x + a_2x^2 + \ldots + a_rx^r = 0$, then A is invertible if $a_0 \neq 0$ and its inverse is given by:

$$A^{-1} = \frac{1}{a_0}[a_1\,I + a_2\,A + \ldots a_r\,A^{r-1}]$$

10.4 Characteristic Equation : (Not explicitly asked in any exam)

Let A be any square matrix of order n, its characteristic equation is a polynomial of degree n given by

$|A - \lambda I_n| = 0$, where I_n denote identity matrix of order n.

$$|A - \lambda I_n| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda \ldots & a_{3n} \\ — & — & — & — \\ — & — & — & — \\ a_{n1} & a_{n2} & a_{n3} \ldots & a_{nn} - \lambda \end{vmatrix} = 0$$

Let A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then its characteristic equation is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=0 \Rightarrow \lambda^2-(a+d)\lambda+ad-bc=0$

Let A be 3×3 matrix, $A=(a_{ij})_{3\times3}$, its characteristic equation is $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}=0$

Roots of characteristic equation are known as characteristic roots of matrix or eigen values.

Result : Every matrix satisfies its characteristic equation.

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its characteristic equation is $\lambda^2-(a+d)\lambda+(ad-bc)=0$, then $A^2-(a+d)A+(ad-bc)I_2=0$

Illustrating the Concepts :

1. If $A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$,

If $A^3=xA^2+yA+zI_3$, Find (x, y, z).

Characteristic equation A is given by

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ p & q & r-\lambda \end{vmatrix}=0 \Rightarrow -\lambda\left(-r\lambda+\lambda^2-q\right)-(-p)=0$$

$\Rightarrow \quad -\lambda^3+r\lambda^2+q\lambda+p=0 \quad \Rightarrow \quad \lambda^3-r\lambda^2-q\lambda-p=0$

$\Rightarrow \quad A^3-rA^2-qA-pI_3=0 \quad \Rightarrow \quad A^3=rA^2+qA+pI_3$

$(x, y, z)\equiv(r, q, p)$

2. If $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $A^{-1}=xA^2+yA+zI$, find (x, y, z).

Characteristic equation of A is : $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix}=0$

$\Rightarrow \quad (1-\lambda)\left((1-\lambda)(4-\lambda)+2\right)=0 \quad \Rightarrow \quad (1-\lambda)\left(\lambda^2-5\lambda+6\right)=0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-11\lambda+6=0 \quad \Rightarrow \quad \lambda^3-6\lambda^2+11\lambda-6=0$

$\Rightarrow \quad A^3-6A^2+11A-6I=0$

$\Rightarrow \quad A^{-1}=\frac{1}{6}\left(A^2-6A+11I\right) \Rightarrow (x, y, z)\equiv\left(\frac{1}{6}, -1, \frac{11}{6}\right)$

ELEMENTARY OPERATIONS (OR TRANSFORMATIONS) Section - 11

An elementary operation on a matrix is an operation which when applied, does not change the order of the matrix. If the operation is applied to rows, then it is called an elementary row operation and when applied to columns, it is called an elementary column operation.

11.1 Elementary Row Operations

The following type of operations are called elementary row operations.

(i) Interchange of two rows: The interchange of i^{th} and j^{th} rows is denoted by R_{ij}, which means $R_i \leftrightarrow R_j$.

(ii) Multiplication of the elements of any row by a non-zero scalar k: Multiplication of the elements of i^{th} row by $k\ (\neq 0)$ is denoted by $R_i\,(k)$, which means $R_i \to kR_i$.

(iii) Addition to the elements of any row of the matrix, corresponding elements of any other row multiplied by a scalar k.
Addition of k times the j^{th} row to the i^{th} row is denoted by $R_{ij}\,(k)$, which means $R_i \to R_i + kR_j$.

11.2 Inverse of a Matrix from Elementary Row Transformation

If A is reduced to I by elementary row (L.H.S.) transformation, then suppose I is reduced to P (R.H.S.) and no change in A in R.H.S.

i.e., $A = IA$

After transformation $I = PA$ then P is the inverse of A $P = A^{-1}$

Illustrating the Concepts :

Use the method of elementary row transformations to compute the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$\Rightarrow$ Write $A = IA$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to -R_2$ and $R_3 \to \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} A$$

Operate $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -\frac{5}{3} & 1 & \frac{1}{3} \end{bmatrix} A$$

Operate $R_3 \to -\frac{1}{7} R_3$

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

Operate $R_1 \to R_1 + 13R_3$ and $R_2 \to R_2 - 9R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS — Section - 12

12.1 Solution of Simultaneous Linear Equations

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, \ldots, x_n$ as given below :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \qquad \ldots \text{(i)} \\ \ldots \quad \ldots \quad &\ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

If $b_1 = b_2 = \ldots = b_n = 0$, then the system of equations (i) is called a system of homogeneous linear equations and if atleast one of $b_1, b_2, \ldots, b_n$ is non-zero, then it is called a system of non-homogeneous linear equations.

We write the above system of equations (i) in the matrix form

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots a_{1m}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{nl}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots a_{nn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & \ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B \qquad \ldots\ldots\text{(ii)}$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & \ldots\ldots & a_{nn} \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \; \textit{and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Pre-Multiplying (ii) by A^{-1}

$\therefore \quad A^{-1}AX = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$

12.2 Rule for Solving System of Equation by using Matrix Method

(A) When system of equations is non-homogeneous :

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1} B$.

(ii) If $|A| = 0$ and $(adj\ A).\ B \neq 0$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(adj\ A).\ B = 0$ then the system of equations is consistent and has an infinite number of solutions.

(B) When system of equations is homogeneous :

(i) If $|A| \neq 0$, the system of equations have only trivial solutions and it has one solution.

(ii) If $|A| = 0$, the system of equations has non-trivial solutions and it has infinite solutions.

(iii) If Number of equations < Number of unknowns, then it has non trivial solution.

Note : Non-homogeneous linear equations can also be solved by Cramer's rule. This method has been discussed in the section on determinants.

Illustrating the Concepts :

Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{pmatrix}$ *and hence solve the equations :*

$x + 2y + 3z = 11, \quad x - 2y + 3z = 3, \quad x + 2y - 3z = -1.$

We have $|A| = 1(6-6) - 2(-3-3) + 3(2+2) = 24$

Now as $|A| \neq 0$, then it is a non-singular matrix hence A^{-1} exist and has unique solution :

Now $adj\ A = \begin{pmatrix} 0 & 12 & 12 \\ 6 & -6 & 0 \\ 4 & 0 & -4 \end{pmatrix}$

$\therefore \quad A^{-1} = \dfrac{1}{|A|} \cdot adj\ A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & -1/4 & 0 \\ 1/6 & 0 & -1/6 \end{pmatrix}$

The solution of the given equations is $X = A^{-1}B$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & -1/4 & 0 \\ 1/6 & 0 & -1/6 \end{pmatrix} \begin{pmatrix} 11 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Hence $x = 1$, $y = 2$ and $z = 2$ is the required solution.

Illustration - 12

The value of $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ *or* $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ *is :*

(A) $(a-b)(b-c)(c-a)$ **(B)** $(b-a)(c-b)(c-a)$
(C) $a(b-c)(c-a)$ **(D)** None of these

SOLUTION : (A)

By property 1, i.e. changing rows into columns, we get the second form of determinant :

Operating $R_2 \to R_2 - R_1$

and $R_3 \to R_3 - R_1$ in first one, we get :

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$= (b-a)\ (c-a)\ [(c+a)-(b+a)]$

$= (b-a)(c-a)(c-b)$

$= (a-b)(c-a)(b-c) =$ R.H.S.

Illustration - 13

The value of $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$ *is :*

(A) $4abc$ **(B)** $3a^2b^2c^2$ **(C)** $4a^2b^2c^2$ **(D)** *None of these*

SOLUTION : (C)

Multiply C_1 by a, C_2 by b and C_3 by c and hence divide the determinant by abc.

$$= \frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & ab^2 & ac^2 \\ a^2b & b(c^2+a^2) & bc^2 \\ a^2c & cb^2 & c(a^2+b^2) \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & b^2 & c^2 \\ a^2 & c^2+a^2 & c^2 \\ a^2 & b^2 & a^2+b^2 \end{vmatrix} \quad \ldots.(i)$$

(Taking out a, b, c from R_1, R_2, R_3 respectively)

Now applying $C_1 \to C_1 - C_2 - C_3$

$$= \begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2+a^2 & c^2 \\ -2b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & c^2+a^2 & c^2 \\ b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= -2\begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & a^2 & 0 \\ b^2 & 0 & a^2 \end{vmatrix}$$

$= -2\,[0 - b^2(c^2a^2 - 0) + c^2(0 - a^2b^2)]$

$= 4\,a^2b^2c^2 =$ RHS

Note : From step (i), you can proceed the approach followed for INE-A, Q.3(iii)

Illustration - 14

The value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3A - B$ *then the value of A and B are :*

(A) $A = 2abc,\ B = a + b + c$ **(B)** $A = 0,\ B = a^2 + b^2 + c$

(C) $A = 3abc,\ B = a + b + c$ **(D)** $A = 3abc,\ B = a^3 + b^3 + c^3$

SOLUTION : (D)

Operating $R_1 \to R_1 + R_2 + R_3$, we get :

$$= \begin{vmatrix} a+b+c & b+c+a & c+a+b \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Operating $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get :

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$$= (a+b+c)\,[(b-c)(a-b) - (c-a)^2]$$

$$= (a+b+c)\,[ab + bc + ca - a^2 - b^2 - c^2]$$

$$= 3\,abc - a^3 - b^3 - c^3 = \text{RHS}$$

Illustration - 15

The value of $\begin{vmatrix} -bc & ca+ab & ca+ab \\ ab+bc & -ca & ab+bc \\ bc+ca & bc+ca & -ab \end{vmatrix}$ *is :*

(A) $\sum ab$ **(B)** $\dfrac{\sum ab}{2}$ **(C)** $(\sum ab)^3$ **(D)** *None of these*

SOLUTION : (C)

Operating $R_1 \to R_1 + R_2 + R_3$, we get :

$$= \begin{vmatrix} \sum ab & \sum ab & \sum ab \\ ab+bc & -ca & ab+bc \\ bc+ca & bc+ca & -ab \end{vmatrix}$$

$$= \sum ab \begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ca & ab+bc \\ bc+ca & bc+ca & -ab \end{vmatrix}$$

Operating $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get :

$$= \sum ab \begin{vmatrix} 0 & 0 & 1 \\ \sum ab & -\sum ab & ab+bc \\ 0 & \sum ab & -ab \end{vmatrix}$$

$$= \left(\sum ab\right)^3 = \text{RHS}$$

Illustration - 16

The value of $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = abc\,(a+b+c)^3$. *The value of is :*

(A) 1 **(B)** 2 **(C)** 5 **(D)** 1

SOLUTION : (B)

Operating $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get :

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ b^2 - (c+a)^2 & (c+a)^2 - b^2 & b^2 \\ 0 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 0 & c-a-b & (a+b)^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - (R_1 + R_2)$, we get :

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 2(a-b) & -2a & 2ab \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ a-b & -a & ab \end{vmatrix}$$

Operating $C_1 \to C_1 + C_2$, we get :

$$= 2(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -b & -a & ab \end{vmatrix}$$

Operating $C_1 \to C_1 + \frac{C_3}{a}$ and $C_2 \to C_2 + \frac{C_3}{b}$, we get:

$$= 2(a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & ab \end{vmatrix}$$

Now evaluating by third row, we get :

$$2(a+b+c)^2 [ab(c^2 + ac + cb)]$$

$$= 2abc(a+b+c)^3 = \text{R.H.S.}$$

Illustration - 17 *The solution of system of equations : $x + 2y + z = 2$, $2x - 3y + 4z = 1$, $3x + 6y + 3z = 6$ have x, y and z is :*

(A) $x = \frac{8+11k}{7}, y = \frac{3+2k}{7}, z = k$ **(B)** $x = \frac{8-11k}{7}, y = \frac{3+2k}{7}, z = k$

(C) $x = \frac{11k}{7}, y = \frac{3+4k}{7}, z = k$ **(D)** *None of these*

SOLUTION : (B)

1^{st} and 3^{rd} equations are integral multiple of each other.

(dependent equations)

$\Rightarrow$ $D = D_1 = D_2 = D_3 = 0$

$\Rightarrow$ infinite solutions

consider $x + 2y + z = 2$

$2x - 3y + 4z = 1$

Let $z = k$

$$\Rightarrow \begin{cases} x + 2y = 2 - k \\ 2x - 3y = 1 - 4k \end{cases}$$

$\Rightarrow$ $y = \frac{3+2k}{7}$ and $x = \frac{8-11k}{7}$

Hence $x = \frac{8-11k}{7}, y = \frac{3+2k}{7}$ and $z = k$

where k is an arbitrary constant.

Illustration - 18 *Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. The value of P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is :*

(A) $P = \begin{bmatrix} 4 & 7 & 7 \\ 3 & 5 & 0 \end{bmatrix}$ **(B)** $P = \begin{bmatrix} 0 & 1 & 7 \\ 3 & 5 & 1 \end{bmatrix}$ **(C)** $P = \begin{bmatrix} 1 & -1 & -7 \\ 3 & -5 & -1 \end{bmatrix}$ **(D)** $P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

SOLUTION : (D)

Given $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Pre-multiplying both sides by B^{-1}

$$B^{-1} BPA = B^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow IPA = B^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow PA = B^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{. . . (i)}$$

To find B^{-1}:

Now $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

$|B| = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1 \neq 0$. As $|B| \neq 0$

so it is a non-singular matrix and hence inverse of B exists.

$\Rightarrow \quad B^{-1} = \dfrac{Adj.B}{|B|} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

Note : For a 2 × 2 matrix, adjoint can be obtained by swapping diagonal elements and changing the sign of non-diagonal elements

Now from (i),

$PA = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow \quad PA = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$

Post-multiplying both sides by A^{-1}

$PAA^{-1} = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} A^{-1}$

$\Rightarrow \quad PI = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} A^{-1}$

$\therefore \quad P = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} A^{-1}$ . . . (ii)

To find A^{-1} :

since $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. Now $|A| = -1 \neq 0$

$\Rightarrow$ it is non-singular matrix and hence A^{-1} exists

$adj.(A) = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$

$A^{-1} = \dfrac{Adj.A}{|A|} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

Now using (ii),

$P = \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$\Rightarrow \quad P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

Illustration - 19

If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. *Show that* $A^2 - 4A - 5I = O$ *where* I *and* O *are the unit matrix and the null matrix of order 3 respectively. Use this result to the value of* A^{-1} *is* :

(A) $\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2/5 & 3/5 & 3/5 \\ 3/5 & -2/5 & 3/5 \\ 3/5 & 3/5 & -2/5 \end{bmatrix}$ **(C)** $\begin{bmatrix} -1 & 3/5 & 3/5 \\ 3/5 & -1 & 3/5 \\ 3/5 & 3/5 & -1 \end{bmatrix}$ **(D)** *None of these*

SOLUTION : (A)

Given :

$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$A^2 = A.A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$

$\Rightarrow \quad A^2 - 4A - 5I$

$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore \quad A^2 - 4A - 5I = 0 \quad \Rightarrow \quad 5I = A^2 - 4A$

Multiply A^{-1} on both sides, we get :

$$5A^{-1} = A - 4I = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

Illustration - 20

If $A = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix}$ *and* $B = \begin{bmatrix} \cos^2\phi & \cos\phi \sin\phi \\ \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}$

And AB is the zero matrix if θ *and* ϕ *differ by*

(A) *an odd multiple of* π **(B)** *an even multiple of* π

(C) *an odd multiple of* $\pi/2$ **(D)** *None of these*

SOLUTION : (C)

Here, $AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi + \sin^2\phi\sin\theta\cos\theta \\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\sin\phi\cos\phi & \cos\theta\cos\phi\sin\theta\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix} = \cos(\theta-\phi)\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

Clearly AB is the zero matrix if $\cos(\theta - \phi) = 0$, *i.e.*, if $\theta - \phi$ is an odd multiple of $\pi/2$.

Illustration - 21

The inverse of the matrix $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *when* $\alpha\delta - \beta\gamma \neq 0$ *is :*

(A) $\begin{bmatrix} \dfrac{\delta}{\alpha\delta-\beta\gamma} & \dfrac{-\beta}{\alpha\delta-\beta\gamma} \\ \dfrac{-\gamma}{\alpha\delta-\beta\gamma} & \dfrac{\alpha}{\alpha\delta-\beta\gamma} \end{bmatrix}$ **(B)** $\begin{bmatrix} \dfrac{\beta}{\alpha\beta-\gamma\delta} & \dfrac{\alpha}{\alpha\delta-\beta\delta} \\ \dfrac{-\alpha}{\alpha\delta-\beta\gamma} & \dfrac{\alpha}{\alpha\beta-\beta\gamma} \end{bmatrix}$

(C) $\begin{bmatrix} \dfrac{\gamma}{\alpha\beta-\gamma\delta} & \dfrac{-\alpha}{\alpha\delta-\beta\delta} \\ \dfrac{\gamma}{\alpha\delta-\beta\gamma} & \dfrac{\beta}{\alpha\delta-\beta\gamma} \end{bmatrix}$ **(D)** *None of these*

SOLUTION : (A)

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$|A| = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma \neq 0 \quad \text{(Given)}$$

i.e. 'A' is non singular.

$\therefore$ A possesses the inverse A^{-1}

$$adj.\, A = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}' \quad \text{(Replacing each element by its co-factor in } A)$$

$$= \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

Now $A^{-1} = \dfrac{adj.A}{|A|} = \dfrac{1}{|A|}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{\delta}{|A|} & \dfrac{-\beta}{|A|} \\ \dfrac{-\gamma}{|A|} & \dfrac{\alpha}{|A|} \end{bmatrix} = \begin{bmatrix} \dfrac{\delta}{\alpha\delta - \beta\gamma} & \dfrac{-\beta}{\alpha\delta - \beta\gamma} \\ \dfrac{-\gamma}{\alpha\delta - \beta\gamma} & \dfrac{\alpha}{\alpha\delta - \beta\gamma} \end{bmatrix}$$

Illustration - 22

If $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, *then find a, b.*

(A) $a = \tan\theta, b = \cot\theta$ **(B)** $a = \cos 2\theta, b = \sin 2\theta$

(C) $a = \cot\theta, b = \tan\theta$ **(D)** $a = \sin\theta, b = \cos 2\theta$

SOLUTION : (B)

$$\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \frac{1}{1 + \tan^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \cos^2\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$= \cos^2\theta \begin{bmatrix} 1 - \tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1 - \tan^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \quad \Rightarrow \quad a = \cos 2\theta,\ b = \sin 2\theta$$

Illustration - 23

If a, b and c are all non-zero such that $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0$, *then the matrix*

$$A = \begin{bmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{bmatrix}$$ *is _________ and _________.*

(A) *symmetric, non-singular* **(B)** *symmetric, singular*

(C) *skew symmetric, singular* **(D)** *skew symmetric, non-singular*

SOLUTION : (A)

Note that A is symmetric. Next, we have $|A| = abc \begin{vmatrix} 1 + \dfrac{1}{a} & \dfrac{1}{b} & \dfrac{1}{c} \\ \dfrac{1}{a} & 1 + \dfrac{1}{b} & \dfrac{1}{c} \\ \dfrac{1}{a} & \dfrac{1}{b} & 1 + \dfrac{1}{c} \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$, we get :

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix} = abc\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix} \qquad \left[\because \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0\right]$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get : $abc\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = abc \neq 0$

Hence A is non-singular.

Illustration - 24 *If $\omega \neq 1$ is a cube root of unity, then*

$$A = \begin{bmatrix} 1+2\omega^{100}+\omega^{200} & \omega^2 & 1 \\ 1 & 1+2\omega^{100}+\omega^{200} & \omega \\ \omega & \omega^2 & 2+\omega^{100}+2\omega^{200} \end{bmatrix} \text{ is :}$$

(A) *non-singular* **(B)** *singular*

(C) *can-not be determined* **(D)** *None of these*

SOLUTION : (B)

We have, $\omega^{3n+1} = \omega$ and $\omega^{3n+2} = \omega^2$

$$A = \begin{bmatrix} 1+2\omega+\omega^2 & \omega^2 & 1 \\ 1 & 1+\omega^2+2\omega & \omega \\ \omega & \omega^2 & 2+\omega+2\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix} \qquad [\because 1+\omega+\omega^2=0]$$

$$|A| = \omega\begin{vmatrix} \omega & \omega & 1 \\ 1 & 1 & \omega \\ \omega & \omega & -\omega \end{vmatrix} = 0 \quad \text{[taking } \omega \text{ common from } C_2]$$

Thus, $|A| = 0$ and hence A is singular.

Illustration - 25 *The matrices X that commute with the matrix* $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *is :*

(A) $X = \frac{1}{2}\begin{pmatrix} 2a & 2b \\ 3b & 2a+3b \end{pmatrix}$ **(B)** $X = \frac{1}{2}\begin{pmatrix} 2b & 2a \\ 3a & 2a+3b \end{pmatrix}$

(C) $X = \frac{1}{3}\begin{pmatrix} 2a+3b & 2a \\ 3a & 2a+3b \end{pmatrix}$ **(D)** *None of these*

SOLUTION : (A)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix that commute with $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$

$\Rightarrow \quad a + 3b = a + 2c, \quad 2a + 4b = b + 2d$

$\Rightarrow \quad 3b = 2c, 2a + 3b = 2d$ . . . (i)

and $\quad c + 3d = 3a + 4c, \quad 2c + 4d = 3b + 4d$

$\Rightarrow \quad 3b = 2c, 3a + 3c = 3d \quad \Rightarrow \quad a + c = d\ ; 3b = 2c$

Thus, A may be taken as $\begin{pmatrix} a & b \\ 3(b/2) & a + 3(b/2) \end{pmatrix}$

$\Rightarrow \quad A = \dfrac{1}{2}\begin{pmatrix} 2a & 2b \\ 3b & 2a + 3b \end{pmatrix}$

where a, b are arbitrary numbers.

Illustration - 26 *If M is a 3×3 matrix, where $M^T M = I$ and det $(M) = I$, then the value of det $(M - I)$ is :*

(A) -1 **(B)** 1 **(C)** 0 **(D)** *None of these*

SOLUTION : (C)

$(M - I)^T = M^T - I = M^T - M^T M = M^T (I - M)$

$\Rightarrow \quad |(M - I)^T| = |M - I| = |M^T||I - M| = |I - M|$

Hence $|M - I| = -|M - I| \quad \Rightarrow \quad |M - I| = 0$

Another Approach :

$\det(M - I) = \det(M - I)\det(M^T)$

$= \det(MM^T - M^T)$

$= \det(I - M^T) = -\det(M^T - I)$

$= -\det(M - I)^T = -\det(M - I)$

$\Rightarrow \quad \det(M - I) = 0$

Illustration - 27 *The solution of the following equations :*

$$\lambda x + 2y - 2z - 1 = 0,\ 4x + 2\lambda y - z - 2 = 0,\ 6x + 6y + \lambda z - 3 = 0$$

considering specially the case when $\lambda = 2$ is :

(A) $x = k,\ y = \dfrac{1}{2} - k,\ z = 0$ **(B)** $x = 2k,\ y = 2 - k,\ z = 1$

(C) $x = k,\ y = 3 - k,\ z = 0$ **(D)** $x = 2k,\ y = \dfrac{1}{2} - k,\ z = 1$

SOLUTION : (A)

$$D = \begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} = 2(\lambda - 2)(\lambda^2 + 2\lambda + 15)$$

$$D_1 = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix} = 2(\lambda + 6)(\lambda - 2)$$

$$D_2 = \begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix} = (\lambda - 2)(2\lambda + 3)$$

$$D_3 = \begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix} = 6(\lambda - 2)^2$$

For $\lambda \neq 2$

$$x = \frac{(\lambda + 6)}{(\lambda^2 + 2\lambda + 15)},\ y = \frac{2\lambda + 3}{2(\lambda^2 + 2\lambda + 15)},$$

$$z = \frac{3(\lambda - 2)}{\lambda^2 + 2\lambda + 15}$$

For $\lambda = 2\ ; D = D_1 = D_2 = D_3 = 0$

Hence infinite solutions exist.

Equations are :

$2x + 2y - 2z = 1$ . . . . (i)

$4x + 4y - z = 2$ . . . . (ii)

$6x + 6y + 2z = 3$ . . . . (iii)

Operate (i) + (ii) – (iii) to get : $z = 0$

Let $x = k$, then $y = \dfrac{1}{2} - k$

Hence $x = k,\ y = \dfrac{1}{2} - k$ and $z = 0$ is the solution where k is any arbitrary constant.

Illustration - 28 *A trust fund had Rs.50,000 that is to be invested into two types of bonds. The first bond pays 5% simple interest per year and the second bond pays 6% simple interest per year. Divide Rs.50,000 among the two types of bonds so as to obtain an annual total interest of Rs.2780.*

(A) Rs.22,000, Rs.28,000 (B) Rs.15,000, Rs.35,000
(C) Rs.30,000, Rs.20,000 (D) Rs.25,000, Rs.25,000

SOLUTION : (A)

Let Rs. 50,000 be invested into two parts Rs. x and Rs. $(50,000 - x)$ in which first part is invested in first type of bond and second part is invested in second type of bond. These two amounts can be written in the form of a row matrix *i.e.*, $A = [x \quad 50,000 - x]$

and the interest per rupee annually for the two bonds are 5/100 and 6/100 which can be written in the form of a column matrix, *i.e.*, $B = \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}$

$\therefore$ Total interest per year

$$= AB = [x \quad 50,000 - x] \times \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}$$

$$= \left[\frac{5}{100}x + \frac{(50,000 - x)\,6}{100}\right]$$

$$= \left[3000 - \frac{x}{100}\right] = [2780] \qquad \text{(Given)}$$

On comparing, $3000 - \dfrac{x}{100} = 2780$

$\Rightarrow \quad 220 - \dfrac{x}{100} = 0 \quad \Rightarrow \quad x = 22,000$

Hence the required amounts are

Rs, 22,000 and Rs. (50,000 – 22,000)

i.e., Rs. 22,000 and Rs. 28,000.

Illustration - 29 *If S is a skew-symmetric matrix of order n and I + S is non-singular, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix of order n.*

(A) *False* (B) *True* (C) *Can not be determined*

SOLUTION : (B)

$$A^T = \left[(I + S)^T\right]^{-1} [I - S]^T$$

$$= (I - S)^{-1} (I + S),$$

Since $S^T = -S$; S being skew-symmetric.

$\therefore \quad A^TA = (I - S)^{-1} (I + S)(I - S)(I + S)^{-1}$

$= (I - S)^{-1} (I - S)(I + S)(I + S)^{-1}$,

Since $(I + S)(I - S) = (I - S)(I + S)$

$= I \,.\, I = I$

$\therefore \; A = (I - S)(I + S)^{-1}$ is a square matrix of order n.

Illustration - 30 *The values of 'α' do the following equations $x + y + z = 1$, $x + 2y + 4z = \alpha$, $x + 4y + 10z = \alpha^2$ have a solution?*

(A) $\alpha = 0, 1$ (B) $\alpha = 1, 2$ (C) $\alpha = 2, 3$ (D) *None of these*

SOLUTION : (B)

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{vmatrix} = 0 \qquad \because R_2 \to R_2 - R_1,\; R_3 \to R_3 - R_1$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 2 & 4 \\ \alpha^2 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \alpha - 2 & -2 & 4 \\ \alpha^2 - 4 & -6 & 10 \end{vmatrix} = 2(\alpha - 1)(\alpha - 2) \qquad \because C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 4 \\ 1 & \alpha^2 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha - 1 & 3 \\ 1 & \alpha^2 - 1 & 9 \end{vmatrix} = -3(\alpha - 1)(\alpha - 2)$$ $\because\ C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & \alpha \\ 1 & 4 & \alpha^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & \alpha - 2 \\ 1 & 3 & \alpha^2 - 4 \end{vmatrix} = (\alpha - 2)(\alpha - 1)$$ $\because\ C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

As $D = 0$, the system is consistent if and only if it has infinite solutions.

$\Rightarrow \quad D_1 = D_2 = D_3 = 0 \quad \Rightarrow \quad \alpha = 1$ or $\alpha = 2$

Case I : $\alpha = 1$

The given equations are :

$x + y + z = 1$ .... (i)

$x + 2y + 4z = 1$ .... (ii)

$x + 4y + 10z = 1$ .... (iii)

Solve equations (i), (ii) and (iii) to get : $y = -3z$ and $x = 1 + 2z$

Hence the solution set is : $x = 1 + 2k,\ y = -3k,\ z = k$ where k is any arbitrary constant.

Case II : $\alpha = 2$

The given equations are :

$x + y + z = 1$ .... (iv)

$x + 2y + 4z = 2$ .... (v)

$x + 4y + 10z = 4$ .... (vi)

Solve equations (iv), (v) and (vi) to get : $x = 2z$ and $y = 1 - 3z$

Hence the solution set is : $x = 2t,\ y = 1 - 3t,\ z = t$ where t is any arbitrary constant.

Illustration - 31 *The inverse of* $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ *is X where A, C are non-singular matrix and also the inverse of* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ *is Y, then X and Y are :*

(A) $X = \begin{bmatrix} A^{-1} & 1 \\ C^{-1}BA^{-1} & C \end{bmatrix},\ Y = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(B) $X = \begin{bmatrix} A & 1 \\ C^{-1}BA^{-1} & C \end{bmatrix},\ Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(C) $X = \begin{bmatrix} A & 0 \\ C^{-1}A^{-1}B & 1 \end{bmatrix},\ Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(D) $X = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix},\ Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

SOLUTION : (D)

First Part :

As $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}$

$$= \begin{pmatrix} AA^{-1} & 0 \\ BA^{-1} - CC^{-1}BA^{-1} & CC^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and $\begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$

$$= \begin{pmatrix} A^{-1}A & 0 \\ -C^{-1}BA^{-1}A + C^{-1}B & C^{-1}C \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Hence $\begin{pmatrix} A^{-1} & 0 \\ C^{-1}BA^{-1} & C^{-1} \end{pmatrix}$ is the inverse of $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = I$.

Second Part :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

Where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Inverse of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

since $C^{-1} BA^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Illustration - 32 *Let A and B be matrices of order n. if $(I - AB)$ is invertible, and $(I - BA)$ is also invertible then the $(I - BA)^{-1}$ value of is :*

(A) I (B) $I - B(I - AB)^{-1}A$ (C) $I + B(I - AB)^{-1}A$ (D) *None of these*

SOLUTION : (C)

$I - BA = BIB^{-1} - BABB^{-1}$

$= B\,(I - AB)\,B^{-1}$ . . . (i)

Hence, $|I - BA| = |B|\,|I - AB|\,|B^{-1}|$

$= |I - AB|\,|B|\,|B^{-1}|$

$= |I - AB|\,|B|\,|B^{-1}| = |I - AB|$ . . . (ii)

$[\because\ |B|\,|B^{-1}| = |BB^{-1}| = |I| = 1]$

If $I - AB$ is invertible, $|I - AB|$ has to be non-zero.

Hence, $|I - BA| \neq 0$ and therefore $I - BA$ is also invertible

Now $(I - BA)\,\{I + B\,(I - AB)^{-1} A\}$

$= (I - BA) + (I - BA)B\,(I - AB)^{-1}A$

$= (I - BA) + \{B\,(I - AB)\,B^{-1}\}\,B\,(I - AB)^{-1} A$ (Using (i))

$= (I - BA) + B(I - AB)\,(I - AB)^{-1}A$

$= I - BA + BA = I$

Hence, $(I - BA)^{-1} = I + B\,(I - AB)^{-1} A$ . . . (iii)